

ON 2-STAR-PERMUTABILITY IN REGULAR MULTI-POINTED CATEGORIES

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ABSTRACT. 2-star-permutable categories were introduced in a joint work with Z. Janelidze and A. Ursini as a common generalisation of regular Mal'tsev categories and of normal subtractive categories. In the present article we first characterise these categories in terms of what we call star-regular pushouts. We then show that the 3×3 Lemma characterising normal subtractive categories and the Cuboid Lemma characterising regular Mal'tsev categories are special instances of a more general homological lemma for star-exact sequences. We prove that 2-star-permutability is equivalent to the validity of this lemma for a star-regular category.

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INTRODUCTION

The theory of *Mal'tsev categories* in the sense of A. Carboni, J. Lambek and M.C. Pedicchio [6] provides a beautiful example of the way how categorical algebra leads to a structural understanding of algebraic varieties (in the sense of universal algebra). Among regular categories, Mal'tsev categories are characterised by the property of 2-permutability of equivalence relations: given two equivalence relations R and S on the same object A , the two relational composites RS and SR are equal:

$$RS = SR.$$

In the case of a variety of universal algebras this property is actually equivalent to the existence of a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ [20]. In the pointed context, that is when the category has a zero object, there is also a suitable notion of 2-permutability, called “2-permutability at 0” [21]. In a variety this property can be expressed by requiring that, whenever for a given element x in an algebra A there is an element y with $xRyS0$ (here 0 is the unique constant in A), then there is also an element z in A with $xSzR0$. The validity of this property is equivalent to the existence of a binary term $s(x, y)$ such that the identities $s(x, 0) = x$ and $s(x, x) = 0$ hold true [21]. Among regular categories, the ones where the property of 2-permutability at 0 holds true are precisely the *subtractive categories* introduced in [16].

The aim of this paper is to look at regular Mal'tsev and at subtractive categories as special instances of the general notion of 2-star-permutable categories introduced in collaboration with Z. Janelidze and A. Ursini in [9]. This generalisation is achieved by working in the context of a *regular multi-pointed category*, i.e. a regular category equipped with an ideal \mathcal{N} of distinguished morphisms [7]. When

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\mathcal{N} is the class of all morphisms, a situation which we refer to as the *total context*, regular multi-pointed categories are just regular categories, and 2-star-permutable categories are precisely the regular Mal'tsev categories. When \mathcal{N} is the class of all zero morphisms in a pointed category, we call this the *pointed context*, regular multi-pointed categories are regular pointed categories, and 2-star-permutable categories are the regular subtractive categories.

This paper follows the same line of research as in [9] which was mainly focused on the property of 3-star-permutability, a generalised notion which captures Goursat categories in the total context and, again, subtractive categories in the pointed context.

In this work we study two remarkable aspects of the property of 2-star-permutability. First we provide a characterisation of 2-star-permutable categories in terms of a special kind of pushouts (Proposition 2.4), that we call *star-regular pushouts* (Definition 2.2). Then we examine a homological diagram lemma of star-exact sequences, which can be seen as a generalisation of the 3×3 Lemma, whose validity is equivalent to 2-star-permutability. We call this lemma the Star-Upper Cuboid Lemma (Theorem 3.3). The validity of this lemma turns out to give at once a characterisation of regular Mal'tsev categories (extending a result in [11]) and, in the pointed context, a characterisation of those normal categories which are subtractive (this was first discovered in [17]).

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1. STAR-REGULAR CATEGORIES

1.1. Regular categories and relations. A finitely complete category \mathbb{C} is said to be a *regular* category [1] when any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. In a regular category any morphism $f: X \rightarrow Y$ has a factorisation $f = m \cdot p$, where p is a regular epimorphism and m is a monomorphism. The corresponding (regular epimorphism, monomorphism) factorisation system is then stable under pullbacks.

A relation ϱ from X to Y is a subobject $\langle \varrho_1, \varrho_2 \rangle: R \rightarrow X \times Y$. The opposite relation, denoted ϱ° , is given by the subobject $\langle \varrho_2, \varrho_1 \rangle: R \rightarrow Y \times X$. We identify a morphism $f: X \rightarrow Y$ with the relation $\langle 1_X, f \rangle: X \rightarrow X \times Y$ and write f° for the opposite relation. Given another relation σ from Y to Z , the composite relation of ϱ and σ is a relation $\sigma\varrho$ from X to Z . With this notation, we can write the above relation as $\varrho = \varrho_2\varrho_1^\circ$. The following properties are well known (see [5], for instance); we collect them in a lemma for future references.

Lemma 1.1. *Let $f: X \rightarrow Y$ be any morphism in a regular category \mathbb{C} . Then:*

- (a) $ff^\circ f = f$ and $f^\circ ff^\circ = f^\circ$;
- (b) $ff^\circ = 1_Y$ if and only if f is a regular epimorphism.

A kernel pair of a morphism $f: X \rightarrow Y$, denoted by

$$(\pi_1, \pi_2): \text{Eq}(f) \rightrightarrows X,$$

is called an *effective equivalence relation*; we write it either as $\text{Eq}(f) = f^\circ f$, or as $\text{Eq}(f) = \pi_2\pi_1^\circ$, as mentioned above. When f is a regular epimorphism, then f is

the coequaliser of π_1 and π_2 and the diagram

$$\text{Eq}(f) \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} X \xrightarrow{f} Y$$

is called an *exact fork*. In a regular category any effective equivalence relation is the kernel pair of a regular epimorphism.

1.2. Star relations. We now recall some notions introduced in [10], which are useful to develop a unified treatment of pointed and non-pointed categorical algebra. Let \mathbb{C} denote a category with finite limits, and \mathcal{N} a distinguished class of morphisms that forms an *ideal*, i.e. for any composable pair of morphisms g, f , if either g or f belongs to \mathcal{N} , then the composite $g \cdot f$ belongs to \mathcal{N} . An \mathcal{N} -*kernel* of a morphism $f : X \rightarrow Y$ is defined as a morphism $n_f : N_f \rightarrow X$ such that $f \cdot n_f \in \mathcal{N}$ and n_f is universal with this property (note that such n_f is automatically a monomorphism). A pair of parallel morphisms, denoted by $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$, is called a *star*; it is called a monic star, or a *star relation*, when the pair (σ_1, σ_2) is jointly monomorphic.

Given a relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ on an object X , we denote by $\varrho^* : R^* \rightrightarrows X$ the biggest subrelation of ϱ which is a (monic) star. When \mathbb{C} has \mathcal{N} -kernels, it can be constructed by setting $\varrho^* = (\varrho_1 \cdot n_{\rho_1}, \varrho_2 \cdot n_{\rho_1})$, where n_{ρ_1} is the \mathcal{N} -kernel of ϱ_1 . In particular, if we denote the discrete equivalence relation on an object X by $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$, then $\Delta_X^* = (n_{1_X}, n_{1_X})$, where n_{1_X} is the \mathcal{N} -kernel of 1_X .

The *star-kernel* of a morphism $f : X \rightarrow Y$ is a universal star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with the property $f \cdot \sigma_1 = f \cdot \sigma_2$; it is easy to see that the star-kernel of f coincides with $\text{Eq}(f)^* \rightrightarrows X$ whenever \mathcal{N} -kernels exist.

A category \mathbb{C} equipped with an ideal \mathcal{N} of morphisms is called a *multi-pointed category* [10]. If, moreover, every morphism admits an \mathcal{N} -kernel, then \mathbb{C} will be called a *multi-pointed category with kernels*.

Definition 1.2. [10] A regular multi-pointed category \mathbb{C} with kernels is called a *star-regular category* when every regular epimorphism in \mathbb{C} is a coequaliser of a star.

In the total context stars are pairs of parallel morphisms, \mathcal{N} -kernels are isomorphisms, star-kernels are kernel pairs and a star-regular category is precisely a regular category. In the pointed context, the first morphism σ_1 in a star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is the unique null morphism $S \rightarrow X$ and hence a star σ can be identified with a morphism (its second component σ_2). Then, \mathcal{N} -kernels and star-kernels become the usual kernels, and a star-regular category is the same as a normal category [18], i.e. a pointed regular category in which any regular epimorphism is a normal epimorphism.

1.3. Calculus of star relations. The calculus of star relations [9] can be seen as an extension of the usual calculus of relations (in a regular category) to the regular multi-pointed context. First of all note that for any relation $\varrho : R \rightrightarrows X$ we have

$$\varrho^* = \varrho \Delta_X^*.$$

Inspired by this formula, for any relation ϱ from X to an object Y , we define

$$\varrho^* = \varrho \Delta_X^* \quad \text{and} \quad {}^*\varrho = \Delta_Y^* \varrho.$$

Note that associativity of composition yields

$$*(\varrho^*) = (*\varrho)^*$$

and so we can write $*\varrho^*$ for the above.

For any relation σ (from some object Y to Z), the associativity of composition also gives

$$(\sigma^*)\varrho = \sigma(*\varrho),$$

and

$$(\sigma\varrho)^* = \sigma\varrho^*.$$

It is easy to verify that for any morphism $f : X \rightarrow Y$ we have

$$f^* = *f^* \quad \text{and} \quad *f^\circ = *f^\circ*.$$

2. 2-STAR-PERMUTABILITY AND STAR-REGULAR PUSHOUTS

Recall that a finitely complete category \mathbb{C} is called a *Mal'tsev category* when any reflexive relation in \mathbb{C} is an equivalence relation [6, 5]. We recall the following well known characterisation of the regular categories which are Mal'tsev categories:

Proposition 2.1. *A regular category \mathbb{C} is a Mal'tsev category if and only if the composition of effective equivalence relations in \mathbb{C} is commutative:*

$$\text{Eq}(f)\text{Eq}(g) = \text{Eq}(g)\text{Eq}(f)$$

for any pair of regular epimorphisms f and g in \mathbb{C} with the same domain.

There are many known characterisations of regular Mal'tsev categories (see Section 2.5 in [2], for instance, and references therein). The one that will play a central role in the present work is expressed in terms of commutative diagrams of the form

$$\begin{array}{ccc} C & \xrightarrow{c} & A \\ g \downarrow \uparrow t & & f \downarrow \uparrow s \\ D & \xrightarrow{d} & B, \end{array} \quad (1)$$

where f and g are split epimorphisms ($f \cdot s = 1_B$, $g \cdot t = 1_D$), $f \cdot c = d \cdot g$, $s \cdot d = c \cdot t$, and c and d are regular epimorphisms. A diagram of type (1) is always a pushout; it is called a *regular pushout* [4] (alternatively, a *double extension* [15, 13]) when, moreover, the canonical morphism $\langle g, c \rangle : C \rightarrow D \times_B A$ to the pullback $D \times_B A$ of d and f is a regular epimorphism. Among regular categories, Mal'tsev categories can be characterized as those ones where any square (1) is a regular pushout: this easily follows from the results in [4], and a simple proof of this fact is given in [12].

Observe that a commutative diagram of type (1) is a regular pushout if and only if $cg^\circ = f^\circ d$ or, equivalently, $gc^\circ = d^\circ f$. This suggests to introduce the following notion:

Definition 2.2. A commutative diagram (1) is a *star-regular pushout* if it satisfies the identity $cg^{\circ*} = f^\circ d^*$ (or, equivalently, $gc^{\circ*} = d^\circ f^*$).

Diagrammatically, the property of being a star-regular pushout can be expressed as follows. Consider the commutative diagram

$$\begin{array}{ccccc}
 & N_g & & & \\
 & \downarrow n_g & & N_a \cdots \cdots \rightarrow N_x & \\
 & C & & & A \\
 & \downarrow g & & \downarrow n_a & \downarrow n_x \\
 & D & \xrightarrow{d} & D \times_B A & \xrightarrow{f} B \\
 & & \nearrow x & \nearrow y & \\
 & & M & & \\
 & \nearrow p & \nearrow b & \nearrow c & \\
 & & & &
 \end{array}
 \quad (2)$$

where $(D \times_B A, x, y)$ is the pullback of (f, d) , $m \cdot p$ is the (regular epimorphism, monomorphism) factorisation of the induced morphism $\langle g, c \rangle: C \rightarrow D \times_B A$. Then the identity $cg^\circ = ba^\circ$ allows one to identify $cg^{\circ*}$ with the relation $(a \cdot n_a, b \cdot n_a)$, while $f^\circ d = yx^\circ$ says that $f^\circ d^*$ can be identified with the relation $(x \cdot n_x, y \cdot n_x)$. Accordingly, diagram (1) is a star-regular pushout precisely when the dotted arrow from N_a to N_x is an isomorphism. Notice that in the total context the \mathcal{N} -kernels are isomorphisms, so that m is an isomorphism if and only if (1) is a regular pushout, as expected.

The “star-version” of the notion of Mal’tsev category can be defined as follows:

Definition 2.3. [9] A regular multi-pointed category with kernels \mathbb{C} is said to be a *2-star-permutable category* if

$$\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*$$

for any pair of regular epimorphisms f and g in \mathbb{C} with the same domain.

One can check that the equality $\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*$ in the definition above can be actually replaced by $\text{Eq}(f)\text{Eq}(g)^* \leq \text{Eq}(g)\text{Eq}(f)^*$.

In the total context the property of 2-star-permutability characterises the regular categories which are Mal’tsev. In the pointed context this same property characterises the regular categories which are subtractive [16] (this follows from the characterisation of subtractivity given in Theorem 6.9 in [17]).

The next result gives a useful characterisation of 2-star-permutable categories. Given a commutative diagram of type (1), we write $g\langle \text{Eq}(c) \rangle$ and $g\langle \text{Eq}(c)^* \rangle$ for the direct images of the relations $\text{Eq}(c)$ and $\text{Eq}(c)^*$ along the split epimorphism g . The vertical split epimorphisms are such that both the equalities $g\langle \text{Eq}(c) \rangle = \text{Eq}(d)$ and $g\langle \text{Eq}(c)^* \rangle = \text{Eq}(d)^*$ hold true in \mathbb{C} .

Proposition 2.4. *For a regular multi-pointed category with kernels \mathbb{C} the following statements are equivalent:*

- (a) \mathbb{C} is a 2-star-permutable category;
- (b) any commutative diagram of the form (1) is a star-regular pushout.

Proof. (a) \Rightarrow (b) Given a pushout (1) we have

$$\begin{aligned}
 f^\circ d^* &= cc^\circ f^\circ d^* && (\text{Lemma 1.1(2)}) \\
 &= cg^\circ d^\circ d^* && (f \cdot c = d \cdot g) \\
 &= cg^\circ gc^\circ c^* g^\circ && (\text{Eq}(d)^* = g\langle \text{Eq}(c)^* \rangle) \\
 &= cc^\circ cg^\circ g^* g^\circ && (\text{Eq}(g)\text{Eq}(c)^* = \text{Eq}(c)\text{Eq}(g)^* \text{ by Definition 2.3}) \\
 &\leq cc^\circ cg^\circ gg^\circ && (g^* \leq g) \\
 &= cg^\circ. && (\text{Lemma 1.1(1)})
 \end{aligned}$$

Since cg° is the largest star contained in cg° , it follows that $f^\circ d^* \leq cg^\circ$. The inclusion $cg^\circ \leq f^\circ d^*$ always holds, so that $cg^\circ = f^\circ d^*$.

(b) \Rightarrow (a) Let us consider regular epimorphisms $f: X \twoheadrightarrow Y$ and $g: X \twoheadrightarrow Z$. We want to prove that $\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*$. For this we build the following diagram

$$\begin{array}{ccc}
 \text{Eq}(f) & \xrightarrow{c} & g\langle \text{Eq}(f) \rangle \\
 \pi_1 \downarrow & \pi_2 & \downarrow \rho_1 \quad \rho_2 \\
 X & \xrightarrow{g} & Z \\
 f \downarrow & & \\
 Y & &
 \end{array}$$

that represents the regular image of $\text{Eq}(f)$ along g . The relation $g\langle \text{Eq}(f) \rangle = (\rho_1, \rho_2)$ is reflexive and, consequently, ρ_1 is a split epimorphism. By assumption, we then know that the equality

$$(A) \quad \rho_1^\circ g^* = c\pi_1^{\circ*}$$

holds true. This implies that

$$\begin{aligned}
 \text{Eq}(f)\text{Eq}(g)^* &= \pi_2\pi_1^\circ g^\circ g^* \\
 &= \pi_2 c^\circ \rho_1^\circ g^* && (g \cdot \pi_1 = \rho_1 \cdot c) \\
 &= \pi_2 c^\circ c\pi_1^{\circ*} && (A) \\
 &\leq \pi_2 c^\circ c\pi_2^\circ \pi_2\pi_1^{\circ*} && (\Delta_{\text{Eq}(f)} \leq \pi_2^\circ \pi_2) \\
 &= \text{Eq}(g)\pi_2\pi_1^{\circ*} && (\pi_2\langle \text{Eq}(c) \rangle = \text{Eq}(g)) \\
 &= \text{Eq}(g)\text{Eq}(f)^*,
 \end{aligned}$$

where the equality $\pi_2\langle \text{Eq}(c) \rangle = \text{Eq}(g)$ follows from the fact that the split epimorphisms π_2 and ρ_2 induce a split epimorphism from $\text{Eq}(c)$ to $\text{Eq}(g)$. \square

In the total context, Proposition 2.4 gives the characterisation of regular Mal'tsev categories through regular pushouts (see [4] and Proposition 3.4 of [12]), as expected. In the pointed context, condition (b) of Proposition 2.4 translates into the pointed version of the *right saturation* property [9] for any commutative diagram of type (1): the induced morphism $\bar{c}: \text{Ker}(g) \rightarrow \text{Ker}(f)$, from the kernel of g to the kernel of f is also a regular epimorphism. This can be seen by looking at diagram (2), where the \mathcal{N} -kernels now represent actual kernels, so that $\text{Ker}(a) = \text{Ker}(x) = \text{Ker}(f)$.

2.1. The star of a pullback relation. Consider the pullback relation $\pi = (\pi_1, \pi_2)$ of a pair (g, δ) of morphisms as in the diagram

$$\begin{array}{ccc} W \times_D C & \xrightarrow{\pi_2} & C \\ \pi_1 \downarrow & \lrcorner & \downarrow g \\ W & \xrightarrow{\delta} & D. \end{array}$$

The *star of the pullback relation* π is defined as $\pi^* = \pi \Delta_W^*$. It can be described as the universal relation $\nu = (\nu_1, \nu_2)$ from W to C such that $\nu_1 \in \mathcal{N}$ and $\delta \cdot \nu_1 = g \cdot \nu_2$ as in the diagram

$$\begin{array}{ccccc} (W \times_D C)^* & \xrightarrow{\nu_2} & C \\ \searrow n_{\pi_1} & & \downarrow g \\ W \times_D C & \xrightarrow{\pi_2} & C \\ \pi_1 \downarrow & \lrcorner & \downarrow g \\ W & \xrightarrow{\delta} & D, \end{array}$$

ν_1 (curved arrow from $(W \times_D C)^*$ to W)

where n_{π_1} is the \mathcal{N} -kernel of π_1 , $\nu_1 = \pi_1 \cdot n_{\pi_1}$ and $\nu_2 = \pi_2 \cdot n_{\pi_1}$.

By using the composition of relations one has the equalities $\pi = \pi_2 \pi_1^\circ = g^\circ \delta$, so that

$$\pi^* = \pi_2 \pi_1^{\circ*} = g^\circ \delta^*.$$

In the total context, the star of a pullback relation is precisely that pullback relation. In the pointed context, the star of the pullback (relation) of (g, δ) is given by $\pi^* = (0, \ker(g))$.

A morphism $f: X \rightarrow Y$ in a multi-pointed category with kernels is said to be *saturating* [9] when the induced dotted morphism from the \mathcal{N} -kernel of 1_X to the \mathcal{N} -kernel of 1_Y making the diagram

$$\begin{array}{ccc} N_{1_X} & \dashrightarrow & N_{1_Y} \\ n_{1_X} \downarrow & & \downarrow n_{1_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commute is a regular epimorphism. All morphisms are saturating in the pointed context. This is also the case for any *quasi-pointed category* [3], namely a finitely complete category with an initial object 0 and a terminal object 1 such that the arrow $0 \rightarrow 1$ is a monomorphism. As in the pointed case, it suffices to choose for \mathcal{N} the class of morphisms which factor through the initial object 0 . In this case we shall speak of the *quasi-pointed context*. In the total context, any regular epimorphism is saturating. The proof of the following result is straightforward:

Lemma 2.5. [9] *Let \mathbb{C} be a regular multi-pointed category with kernels. For a morphism $f: X \rightarrow Y$ the following conditions are equivalent:*

- (a) f is saturating;
- (b) $\Delta_Y^* = f^* f^\circ$.

The next result gives a characterisation of 2-star-permutable categories which will be useful in the following section.

Proposition 2.6. *For a regular multi-pointed category \mathbb{C} with kernels and saturating regular epimorphisms the following statements are equivalent:*

- (a) \mathbb{C} is a 2-star-permutable category;
(b) for any commutative diagram

$$\begin{array}{ccccc}
 (W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* & & \\
 \downarrow \nu_1 & \searrow \nu_2 & \downarrow \chi_1 & \searrow \chi_2 & \\
 W & \xrightarrow{c} & C & \xrightarrow{c} & A \\
 \downarrow \delta & \searrow g & \downarrow t & \searrow \beta & \downarrow f \\
 D & \xrightarrow{d} & B & & \\
 & & & & \downarrow s
 \end{array}
 \quad (3)$$

where the front square is of the form (1), $\beta \cdot w = d \cdot \delta$, w is a regular epimorphism, $((W \times_D C)^*, \nu_1, \nu_2)$ and $((Y \times_B A)^*, \chi_1, \chi_2)$ are stars of the corresponding pullback relations, then the comparison morphism $\lambda: (W \times_D C)^* \rightarrow (Y \times_B A)^*$ is also a regular epimorphism.

Proof. (a) \Rightarrow (b) To prove that the arrow λ in the cube above is a regular epimorphism, we must show that $\langle \chi_1, \chi_2 \rangle \lambda$ in the commutative diagram

$$\begin{array}{ccc}
 (W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* \\
 \downarrow \langle \nu_1, \nu_2 \rangle & & \downarrow \langle \chi_1, \chi_2 \rangle \\
 W \times C & \xrightarrow{w \times c} & Y \times A
 \end{array}$$

is the (regular epimorphism, monomorphism) factorisation of the morphism $\langle w \cdot \nu_1, c \cdot \nu_2 \rangle: (W \times_D C)^* \rightarrow Y \times A$. That is, we must have $c\nu_2\nu_1^\circ w^\circ = \chi_2\chi_1^\circ$ or, equivalently, $cg^\circ\delta^*w^\circ = f^\circ\beta^*$, since $\nu_2\nu_1^\circ = \nu^* = g^\circ\delta^*$ and $\chi_2\chi_1^\circ = \chi^* = f^\circ\beta^*$ (see Section 2.1).

The front square of diagram (3) is a star-regular pushout by Proposition 2.4, which means that the equality

$$(B) \quad cg^\circ = f^\circ d^*$$

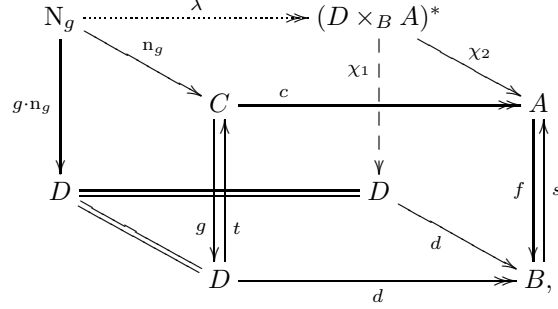
holds true. Now, we always have

$$\begin{aligned}
 cg^\circ\delta^*w^\circ &\leq f^\circ d\delta^*w^\circ && \text{(commutativity of the front face of (3))} \\
 &= f^\circ\beta w^*w^\circ && (d \cdot \delta = \beta \cdot w) \\
 &= f^\circ\beta\Delta_Y^* && \text{(Lemma 2.5)} \\
 &= f^\circ\beta^*.
 \end{aligned}$$

The other inequality follows from

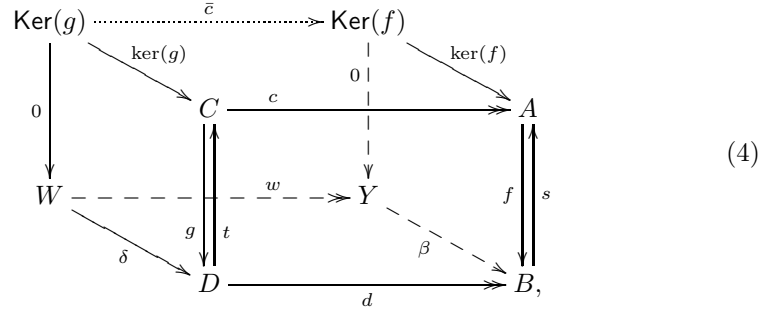
$$\begin{aligned}
 cg^\circ\delta^*w^\circ &\geq cg^{\circ*}\delta^*w^\circ && (g^\circ \geq g^{\circ*}) \\
 &= f^\circ d^*\delta^*w^\circ && (B) \\
 &= f^\circ d\delta^*w^\circ && (*\delta^* = \delta^*; \text{Section 1.3}) \\
 &= f^\circ\beta^*. && \text{(as in the inequality above)}
 \end{aligned}$$

(b) \Rightarrow (a) A commutative diagram of type (1) induces a commutative cube



where $\nu = (g \cdot n_g, n_g)$ is the star of the pullback (relation) of $(g, 1_D)$. By assumption, λ is a regular epimorphism which translates into the equality $cg^\circ 1_D^* 1_D = f^\circ d^*$, as observed in the first part of the proof. We get the equality $cg^\circ = f^\circ d^*$, and this proves that diagram (1) is a star-regular pushout and, consequently, that \mathbb{C} is a 2-star-permutable category by Proposition 2.4. \square

In the total context, Proposition 2.6 is the “star version” of Proposition 3.6 in [12] (see also Proposition 4.1 in [4]). In the pointed context condition (b) of Proposition 2.6 also reduces to the pointed version of the right saturation property (in the sense of [9]). Indeed, in this context that condition says that, in the following commutative diagram



the induced arrow $\bar{c}: \text{Ker}(g) \rightarrow \text{Ker}(f)$ is a regular epimorphism.

We conclude this section with the pointed version of Propositions 2.4 and 2.6:

Corollary 2.7. (see Theorem 2.12 in [9]) *For a pointed regular category \mathbb{C} the following statements are equivalent:*

- (a) \mathbb{C} is a subtractive category;
- (b) any commutative diagram of the form (1) is right saturated, i.e. the comparison morphism $\bar{c}: \text{Ker}(g) \rightarrow \text{Ker}(f)$ is a regular epimorphism.

3. THE STAR-CUBOID LEMMA

In [12] it was shown that regular Mal'tsev categories can be characterised through the validity of a homological lemma called the Upper Cuboid Lemma, a strong form of the denormalised 3×3 Lemma [4, 19, 11]. We are now going to extend this result to the star-regular context. We shall then observe that, in the pointed

context, it gives back the classical Upper 3×3 Lemma characterising subtractive normal categories.

3.1. \mathcal{N} -trivial objects. An object X in a multi-pointed category is said to be \mathcal{N} -trivial when $1_X \in \mathcal{N}$. If a composite $f \cdot g$ belongs to \mathcal{N} and g is a strong epimorphism, then also f belongs to \mathcal{N} . This implies that \mathcal{N} -trivial objects are closed under strong quotients. One says that a multi-pointed category \mathbb{C} *has enough trivial objects* [8] when \mathcal{N} is a closed ideal [14], i.e. any morphism in \mathcal{N} factors through an \mathcal{N} -trivial object and, moreover, the class of \mathcal{N} -trivial objects is closed under subobjects and squares, where the latter property means that, for any \mathcal{N} -trivial object X , the object $X^2 = X \times X$ is \mathcal{N} -trivial. An equivalent way of expressing the existence of enough trivial objects is recalled in the following:

Proposition 3.1. [8] *Let \mathbb{C} be a regular multi-pointed category with kernels. The following conditions are equivalent:*

- (a) *if $(\sigma_1, \sigma_2) : S \rightrightarrows X$ is a relation on X such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$;*
- (b) *\mathbb{C} has enough trivial objects.*

In the following we shall also assume that \mathcal{N} -trivial objects are closed under binary products. Remark that in the total and in the (quasi-)pointed contexts there are enough trivial objects, and \mathcal{N} -trivial objects are closed under binary products.

Under the presence of enough trivial objects the assumption that \mathcal{N} -trivial objects are closed under binary products is equivalent to the following condition:

- (a') *if $(\sigma_1, \sigma_2) : S \rightrightarrows X \times Y$ is a relation from X to Y such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$.*

Whenever the category has enough trivial objects, condition (a') implies that star-kernels “commute” with stars of pullback relations:

Lemma 3.2. *Let \mathbb{C} be a multi-pointed category with kernels, enough trivial objects, and assume that \mathcal{N} -trivial objects are closed under binary products. Given a commutative cube*

$$\begin{array}{ccccc}
 (W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* & & \\
 \downarrow \nu_1 & \searrow \nu_2 & \downarrow \chi_1 & \searrow \chi_2 & \\
 & C & \xrightarrow{c} & A & \\
 & \downarrow g & \downarrow & \downarrow f & \\
 W & \xrightarrow{\delta} & D & \xrightarrow{d} & B \\
 & \uparrow w & \uparrow \beta & & \\
 & Y & & &
 \end{array}$$

in \mathbb{C} , consider the star-kernels of c , d and w , and the induced morphisms $\bar{\delta} : \text{Eq}(w)^* \rightarrow \text{Eq}(d)^*$ and $\bar{g} : \text{Eq}(c)^* \rightarrow \text{Eq}(d)^*$. Then the following constructions are equivalent (up to isomorphism):

- taking the horizontal star-kernel of λ and then the induced morphisms $\text{Eq}(\lambda)^* \rightarrow \text{Eq}(w)^*$ and $\text{Eq}(\lambda)^* \rightarrow \text{Eq}(c)^*$;

- taking the star of the pullback (relation) of \bar{g} and $\bar{\delta}$ and then the induced morphisms $(\text{Eq}(w)^* \times_{\text{Eq}(d)^*} \text{Eq}(c)^*)^* \rightrightarrows (W \times_D C)^*$.

Proof. This follows easily by the usual commutation of kernel pairs with pullbacks and condition (a'). \square

In a star-regular category, a (short) *star-exact sequence* is a diagram

$$\text{Eq}(f)^* \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} X \xrightarrow{f} Y$$

where $\text{Eq}(f)^*$ is a star-kernel of f and f is a coequaliser of f_1 and f_2 (which, by star-regularity, is the same as to say that f is a regular epimorphism). In the total context, a star-exact sequence is just an exact fork, while in the (quasi-)pointed context it is a short exact sequence in the usual sense.

The Star-Upper Cuboid Lemma

Let \mathbb{C} be a star-regular category. Consider a commutative diagram of morphisms and stars in \mathbb{C}

$$\begin{array}{ccccc}
 P & \xrightarrow{\pi} & (W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* \\
 \tau_1 \swarrow & & \nu_1 \swarrow & & \chi_1 \swarrow \\
 \text{Eq}(w)^* & \xrightarrow{\quad} & W & \xrightarrow{w} & Y \\
 \bar{\delta} \swarrow & & \delta \swarrow & & \beta \swarrow \\
 & & \text{Eq}(c)^* & \xrightarrow{\quad} & C \\
 \bar{g} \swarrow & & \nu_2 \swarrow & & \chi_2 \swarrow \\
 S & \xrightarrow{\sigma} & D & \xrightarrow{d} & B,
 \end{array} \quad (5)$$

where the three diamonds are stars of pullback (relations) of regular epimorphisms along arbitrary morphisms (so that $P = (\text{Eq}(w)^* \times_S \text{Eq}(c)^*)^*$) and the two middle rows are star-exact sequences. Then *the upper row is a star-exact sequence whenever the lower row is*.

Note that, in the diagram (5) above, d is necessarily a regular epimorphism, $d \cdot \sigma_1 = d \cdot \sigma_2$ since \bar{g} is an epimorphism, and $\lambda \cdot \pi_1 = \lambda \cdot \pi_2$, because the pair of morphisms (χ_1, χ_2) is jointly monomorphic.

Theorem 3.3. *Let \mathbb{C} be a star-regular category with saturating regular epimorphisms, enough trivial objects, and assume that \mathcal{N} -trivial objects are closed under binary products. The following conditions are equivalent:*

- (a) \mathbb{C} is a 2-star-permutable category;
- (b) the Star-Upper Cuboid Lemma holds true in \mathbb{C} .

Proof. (a) \Rightarrow (b) Suppose that the lower row is a star-exact sequence. The fact that $\pi = \text{Eq}(\lambda)^*$ follows from Lemma 3.2. As explained in Proposition 2.6, λ is a

regular epimorphism if and only if $cg^\circ\delta^*w^\circ \geq f^\circ\beta^*$. In fact we have

$$\begin{aligned}
cg^\circ\delta^*w^\circ &= cc^\circ cg^\circ gg^\circ\delta^*w^\circ && (\text{Lemma 1.1(1)}) \\
&\geq cc^\circ cg^\circ g^*g^\circ\delta^*w^\circ && (\text{Eq}(g) \geq \text{Eq}(g)^*) \\
&= cg^\circ gc^\circ c^*g^\circ\delta^*w^\circ && (\text{Eq}(c)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(c)^*; \text{Definition 2.3}) \\
&= cg^\circ d^\circ d^*\delta^*w^\circ && (g(\text{Eq}(c)^*) = \text{Eq}(d)^* \text{ by assumption}) \\
&= cg^\circ d^\circ d\delta^*w^\circ && (*\delta^* = \delta^*; \text{Section 1.3}) \\
&= cc^\circ f^\circ\beta w^*w^\circ && (d \cdot g = f \cdot c, d \cdot \delta = \beta \cdot w) \\
&= f^\circ\beta w^*w^\circ && (\text{Lemma 1.1(2)}) \\
&= f^\circ\beta\Delta_Y^* && (\text{Lemma 2.5}) \\
&= f^\circ\beta^* && (\text{Section 1.3})
\end{aligned}$$

(b) \Rightarrow (a) Consider a commutative cube of the form (3). We construct a commutative diagram of type (5) by taking the star-kernels of c , w , d and λ , so that \bar{g} , $\bar{\delta}$, τ_1 and τ_2 are the induced arrows between the star-kernels. By Lemma 3.2 we know that (τ_1, τ_2) is the star above the pullback (relation) of $(\bar{g}, \bar{\delta})$. By applying the Star-Upper Cuboid Lemma to this diagram we conclude that the upper row is a star-exact sequence and, consequently, λ is a regular epimorphism. By Proposition 2.6, \mathbb{C} is a 2-star-permutable category. \square

In the total context, Theorem 3.3 is precisely Theorem 4.3 in [12], which gives a characterisation of regular Mal'tsev categories through the Upper Cuboid Lemma, as expected. In the pointed context, the Star-Upper Cuboid Lemma gives the classical Upper 3×3 Lemma: in the pointed version of diagram (5), the back part is irrelevant (like in diagram (4)). Then the front part is a 3×3 diagram where all columns and the middle row are short exact sequences. The Star-Upper Cuboid Lemma claims that the upper row is a short exact sequence whenever the lower row is, i.e. the same as the Upper 3×3 Lemma. The pointed version of Theorem 3.3 is Theorem 5.4 of [18] which characterises normal subtractive categories. Note that in the pointed context, the Upper 3×3 Lemma is also equivalent to the Lower 3×3 Lemma as shown in [18].

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